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AN ADDRESS.*

By E. S. Crawley.

Mr. President, Ladies and Gentlemen: I bring you cordial greetings from the parent Association of which you are forming a Section here, and from your older sister Section in the eastern part of the State. The President of the Association, my colleague, Dr. Schwatt, desired me especially to extend to you his congratulations and best wishes.

When Professor Long invited me to come here to speak at this meeting, he asked me to name my subject. I found it more difficult to do this than to think of what I wanted to say. In fact what I want to do is to speak of a number of things, all connected with mathematical teaching it is true, but not necessarily very closely connected with each other, or forming what could in any sense be called an address upon a set topic. What I shall have to say to you will not be novel. I can do no more than voice what all of you have thought at one time or another, because the experiences which we get from the pursuit of our profession must be pretty much the same everywhere, and our conclusions as to what is good and what is bad, its difficulties, its troubles of various sorts, and its pleasures and satisfactions, must be in the main alike. It is a good thing, nevertheless, for us to meet together and give utterance to our thoughts, even though we realize what I have just said, for it helps to give substantiality to our ideas. Moreover it is helpful to know that others are struggling with the same problems that vex us.

The tendency of teachers to form associations is not a very recent tendency, although some of us who have been in the harness for twenty-five years or more can recall the time when there were no such associations, or when their influence and activity were much more restricted than they have since come to be. The principal object of such associations is the promotion of efficiency. Apart from this they would have little reason for existence. That they accomplish this object is shown by their continued growth, activity and ever-widening influence. I mention one thing in this connection, as it is a most unfortunate condition, but one which happily cannot well continue under the influence of an active and vigorous association such as this: I refer to the fact that there has often been in the past too much lack of sympathy between college teachers and school teachers. It has come about doubtless from the unfortunate relation they hold to each other in that the college teacher receives the product of the school teacher's effort and does not always find it to his liking. Being human he naturally says, or thinks if he doesn't say, unpleasant things about the teacher in the school. Doubtless this censure is sometimes deserved, and doubtless also if the product of the college teacher were judged in the same way by some one higher up, he would likewise often be exposed to well-deserved censure. You have heard the old joke on our friends, the physicians, that their mistakes are all buried, so in a sense is it with the mistakes of the college teacher: they are swallowed up in the hurly-burly of life and it is hard to trace them back to their source. Now, as a matter of fact, the teacher in the school has the harder, and, in some ways the more responsible task, and his failures should be judged very charitably by those who follow after. It is the school teacher's duty to sow the seed, to tend the young shoot, and to start it in the right direction. I wish I had words in which to tell you how strongly I feel the deep and vital importance of these beginnings in the study of mathematics. So much, so very much, depends upon getting the student to have the right point of view. Note that I do not say "giving" the student the right point of view. You cannot give it to him; you can help him to get it, but he *must get it for himself.* Of course, there is the technique, of which I shall speak later, but with the right point of view, or the right orientation with respect to the subject, if you prefer that way of putting it, there will be comparatively little difficulty, in most cases, with the technique.

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* Delivered at the opening of the Pittsburgh Section.

Now a word as to what I mean by getting the right point of view. I mean simply that the student should be led to realize that mathematics is after all nothing but common sense. I am sure that time and again you have all been astounded at the evident helplessness of students, apparently in the full possession of their normal faculties, who have halted and stumbled over the simplest deductions. I believe that this is almost invariably due to their utter failure to realize the common sense of the thing. Now as I have said, it is not enough to tell the student this, however impressively you may do it. He must get to see it for himself. That point attained, however, he is prepared to appreciate the beauty of mathematics, to enjoy its simplicity, and to see how each part helps the other and does its share in rearing the whole structure. In this connection the opening sentences of Dr. A. N. Whitehead's recent book, "Introduction to Mathematics," seem most appropriate and, with your permission, I shall read them. Dr. Whitehead says:

"The study of mathematics is apt to commence in disappointment. The important applications of the science, the theoretical interest of its ideas, and the logical rigour of its methods, all generate the expectation of a speedy introduction to processes of interest. We are told that by its aid the stars are weighed and the billions of molecules in a drop of water are counted. Yet, like the ghost of Hamlet's father, this great science eludes the efforts of our mental weapons to grasp it—''Tis here, 'tis there, 'tis gone'—and what we do see does not suggest the same excuse for illusiveness as sufficed for the ghost, that it is too noble for our gross methods. 'A show of violence,' if ever excusable, may surely be 'offered' to the trivial results which occupy the pages of some elementary mathematical treatises.

"The reason for this failure of the science to live up to its reputation is that its fundamental ideas are not explained to the student disentangled from the technical procedure which has been invented to facilitate their exact presentation in particular instances. Accordingly, the unfortunate learner finds himself struggling to acquire a knowledge of a mass of details which are not illuminated by any general conception. Without a doubt, technical facility is a first requisite for valuable mental activity: we shall fail to appreciate the rhythm of Milton, or the passion of Shelley, so long as we find it necessary to spell the words and are not quite certain of the forms of the individual letters. In this sense there is no royal road to learning. But it is equally an error to confine attention to technical processes, excluding consideration of general ideas. Here lies the road to pedantry."

The next question which naturally presents itself is whether our teachers of mathematics are generally prepared to do this kind of work. I have no doubt that many are doing so; others perhaps need but a hint to set them thinking about it. There are doubtless many others, however, who are content to jog along teaching what the book says, and seeing that the pupils get the right answers without doing anything to vivify the subject or to articulate it with the facts of everyday life, as the current phrase goes. Now I, personally, am not so much concerned with "articulating it with the facts of everyday life"—that will take care of itself—as I am with articulating it with the laws of everyday common sense, by which, to explain further, I mean leading the student to make his mathematical thinking a spontaneous part of his mental activity, just as his thinking about football, the purchase of a new hat, or any other matter which does not require conscious mental effort. Many teachers, however, are helpless in an undertaking of this kind, for they, themselves, lack the necessary foundation, but I am aware that they are often the victims, perhaps the unwilling victims of a bad system. Let us suppose a case: Suppose in a school there is need, in an emergency, for a teacher to take charge of a class in Latin. The head of the school in looking around for someone to fill the gap finds a teacher who has vacant time at the periods when the class in question is to meet and asks him to step in. He demurs on the ground that he has studied very little Latin, that he has never read the author with whom the class is engaged, and that besides this, it has been so many years since he has opened a Latin book that he forgets almost all he ever knew about it. Can you conceive the head of the school insisting that notwithstanding these handicaps the teacher take the class? I will not say that this could not happen, for it is said that all things are possible, and I suppose interlinears are still in the market, but I think you will agree with me that it is very unlikely to happen. Yet I venture to say that when the subject is algebra or geometry, even more, if it be arithmetic, the exact parallel does happen with painful frequency. Yet I maintain that in no subject are specially trained teachers more necessary for successful results than in mathematics. If our associations can open the eyes of principals and school boards to this fact, they will be doing splendid work. To teach from the book, and simply from the book, means inevitable failure in mathematical work. No text-book is entirely satisfactory to the capable teacher. He will sometimes wish to amplify the subject matter, or to present it from a different point of view, and the mere fact of his doing so acts as a stimulus to his students, for they see that the subject is not a mere lot of cut and dried formulæ or theorems between the covers of a book, but a living thing about which one can think, and which one can handle for himself. Again, the capable teacher will never accept as a reason why a thing is so the statement that that is what the book says, or that is what the teacher himself has said. Dogmatic assertions, no matter what the authority, must be banished absolutely from the mathematical class room. This does not mean of course that we may not give our students information about facts the demonstration of which is at the time beyond their reach. Such information is in fact stimulating to an ambitious student as it points the way for him to the higher levels. There is nothing so discouraging because there is nothing so hopeless, as to have a student tell you that he has "finished algebra" or "finished geometry."

Just now I said that when the student has gotten the right point of view of the subject there will not be much difficulty with the technique. Of course it will not be all plain sailing. All that I can do is to point out certain matters which my experience as a teacher have brought to my notice, which lead me often to wish that the earlier training of some of the students I get had been conducted in some way differently. My theorizing as to what might be done may not be of much value, but it may at least form a basis for discussion. Speaking of algebra, the fundamental difficulty with the inefficient student seems often to be failure to comprehend algebraic notation. Algebraic notation so far as it goes constitutes a language, and I often wonder if enough stress is laid upon this phase of the matter. Every algebraic statement (*i. e.*, every algebraic equation or identity) and every algebraic expression can be translated into English, and every statement in words which has to do with suitable matters can be translated into the language of algebra. The former process is usually the easier, at least until a very complete mastery of algebraic technique has been acquired, so that in this respect the relation of algebra to English is the same as in the case of other languages. Now until a student can readily translate any algebraic expression into its equivalent English, or better until he can without such actual translation form from the expression itself a clear mental picture of the meaning, he cannot be said to have gained a satisfactory mastery of algebraic technique. No student who looks at the expression $\frac{1+2x}{3+2x}$ with any appreciation of its meaning will say that we can cancel the 2's and get $\frac{1+x}{3+x}$, and yet there is no more common error with the indifferent, or the ill-prepared student. I wish by the way that the term "cancel" could be eliminated entirely from the vocabulary so far as it applies to the operation of dividing the numerator and denominator of a fraction by the same factor.

While in its expression algebra is appropriately denominated a language, in its operation it is a machine. Herein lies its power. In applying algebra to the solution of a problem we substitute its purely mechanical and easily applied operations for what would require otherwise a complex mental process. Students often, or perhaps generally, fail to appreciate this or only half appreciate it. When I was a boy in school we had once a week an exercise called mental arithmetic, in which we were called upon to solve without so much as a stroke with pencil or chalk a variety of problems by a process of pure reasoning. These problems were usually of a kind that could be solved much more easily, by which I mean with much less conscious mental effort, by algebra; and perhaps the recognition of this fact helped us to appreciate the value of algebra. I understand that such exercises are banished from the curriculum now, but I sometimes wonder if they might not be restored with advantage. Algebra, being a machine, must be correctly used. If one pulls the wrong lever, or turns a crank the wrong way, he will probably get some result, but not the one he should get, and if he is not aware that he has misapplied the machine he will be left in ignorance regarding the reliability of his result. The correct application of the machine requires careful observation and attention, and a habit of precision, all qualities not usually innate, which therefore must be developed. I have now in one of my classes a student who never by any chance seems able to see a minus sign if it is attached to the first term of an expression. Another fault of the same general character which affects multitudes of students is what I term blindness of the left eye. This is an affection not ordinarily manifest, but breaks out, so to speak, only when the student starts to work transforming an equation. He will go through all sorts of evolutions with the right hand side of the equation, changing its value in all sorts of ways, apparently quite blind to the fact that the equation has a left hand side at all. This is especially true when the left side of the equation is a single expression, dy/dx perhaps, whose value he is trying to reduce to a simpler form. Another most desirable quality in the student of algebra, and in fact of any branch of mathematics, is a sense of order. When I see how some students do their work my surprise is not that they sometimes get things wrong, but rather that they ever get them right, so hopelessly lacking in any orderly arrangement is what they produce. One of my colleagues at the University of Pennsylvania has a very forcible way of bringing home to his classes the advantages of order in one's work. He rapidly puts on the black-board twenty-five or thirty crosses arranged in no special order, but hit or miss all jumbled up together, and asks if any of the class can tell at a glance how many there are. Of course no one can. He then arranges the same number of crosses in rectangular order, and the number is at once disclosed. He tells me that this simple expedient has been very successful in impressing the students with the value of order.

Students quite as a matter of course fail to appreciate to anything like its full extent the power of the tool called algebra. I say, quite as a matter of course, because it is natural that this should be so. We as teachers usually expect too much. We forget the enormous advantage which we have through our familiarity with the processes due to the constant repetition of them which we are obliged to make. Until such a facility in handling algebraic expressions has been acquired as to make the operations a matter of second nature, so to speak, one cannot give his entire attention to the real meaning of what he is doing. And in this connection, I believe that the element of time plays a great part. Most of you I think will be able with me to recall experiences where after working unsuccessfully at some problem or discussion without success, and leaving it in disgust, you have come back to it after a week or a month, or perhaps longer and find to your astonishment that there now appears to be no difficulty at all. Your mind has in the meanwhile been unconsciously adjusting itself to the situation and the correct point of view has been reached. So with students time is needed to assimilate the new ideas which the study presents to them, and if it is not given, mental indigestion follows. For this reason I think that in the study of a subject like algebra as much can be accomplished with two periods a week for two years as with five periods a week for one year.

I find amongst the students who come to me two faults, or rather one fault and one error of judgment prominently in evidence. The first is inability to separate what is essential from what is merely incidental in an operation. Very briefly and simply what I mean is this, and I shall use numbers to illustrate my point for they will serve quite well. If a student is asked to multiply 20 by 18 and divide the result by 45 times 8, he often does not realize as fully as he should that when he has written $\frac{20 \times 18}{45 \times 8}$ the operation is actually completed, and that anything he may do subsequently is merely a transformation of the result in a simpler form, that is, it is merely incidental. The second point to which I refer above is a tendency to do a great deal of unnecessary multiplying of algebraic expressions. In most of the operations of algebra, it is better to leave an expression as the expressed product of factors, for then one can see how it is made up, and I constantly find it necessary to warn students not only against the futility, but of the positive harm often done by needlessly performing expressed multiplications.

I find my students as a rule lamentably ignorant of the meaning of the technical language of algebra. Such words as "term," "rational" and "irrational," "imaginary," "root of an equation," "degree of an expression," and various others very often convey no meaning to the student's mind, or at any rate no clearcut definite meaning, so that to make use of such expressions in talking to a class is often sheer waste of time. Another point to mention is a difficulty arising out of the fact that as we advance in the study of mathematics our way of looking at things shades over very gradually from one point of view to another, and the student does not always follow as he should. This statement is very vague and I will try to be more specific.

In the study of elementary algebra, for example, x represents to the student some unknown number, with the emphasis upon the adjective " unknown." But as we advance we change gradually our point of view and begin to think of x as a variable, until finally we learn to think of it exclusively as a variable and the question of whether it is known or unknown is often if not usually of quite minor importance. The introduction of the graph into algebra is an easy and effective way to make this transition. But although graphical methods have now been in vogue in elementary teaching for some time I still find that while students can generally solve a quadratic equation, they are not well acquainted with it as a rational quadratic function of a variable which vanishes for two values of the variable, called the roots of the equation, nor do they readily handle this quadratic expression in the different ways in which it is used, ways which have no connection with its actual solution.

Mathematicians have in recent years been profoundly interested in the study of the fundamentals of both algebra and geometry. While as teachers it is our business to know something of the results of these investigations on account of their great significance and the enlarged view they give us of mathematical operations, we cannot introduce them except with great caution to our younger class of students, and in this class I would include the generality of students in college as well as those in school. They are rather for the selected few. This leads me to make the remark that I deem it of doubtful advantage to bring to a student's attention matters which either through his immaturity of mind or want of preliminary training he is not yet ready to appreciate. It can serve only to befog him and to raise in his mind an impression that, however it may be for others, he has reached the limit of his powers of comprehension of so abstruse a subject. Similarly it is useless to expect students to appreciate refinements of demonstration intended to give scientific completeness, when the special difficulties which such refinements are intended to meet have not occurred to the student himself, or when if pointed out to him he does not grasp their force or significance. On the other hand there is danger in holding off too much from such matters, as reference to them sometimes plants the seed of strong future growth. The wise course to pursue requires judgment and tact. The teacher of elementary algebra encounters a situation of this kind when he takes up imaginaries. To a great many students $\sqrt{-1}$ is bound to remain always merely an "imaginary" or impossible number, but to some it can be revealed in its true significance as the door leading to a new and fruitful field of endeavor, for it leads to a new algebra, a more extended number system, where our old unit of everyday life is not the only unit. Thus the ideas are broadened and made receptive to still wider generalizations to come later.

No more striking results in mathematics have been attained than those which have followed from investigations into the foundations upon which geometry rests. The geometry of the school, however, must remain, as it has been, the geometry of our experience. As a result of these investigations, however, we are more ready than was the past generation of teachers to admit that experience and intuition play a significant part in the study of geometry. We do not insist so much as formerly upon definitions of such concepts as point, right line, and angle, because we realize that the pupil knows by intuition and from experience what they mean. In fact no satisfactory definitions of these and similar concepts have as yet been given, and perhaps never will be given. So also with many propositions. In short, we recognize, or perhaps I would better say we frankly acknowledge, that geometry is not such a perfect bit of logical deduction as we have been inclined in the past to consider it. I would not have you think that I am disparaging the value of geometrical reasoning; far from it; I am only trying to point out that we are willing to readjust our views as to what it is permissible to accept as the foundation upon which to build up our logically derived series of theorems. The difference is that we now know that as we vary the foundation we vary in many respects the superstructure, and that we actually have a choice in the matter of the foundation. It remains true, however, that for the geometry of our high schools the foundation chosen must be that of everyday experience. If you have not read the Provisional Report of the National Committee of Fifteen on Geometry Syllabus, I would advise you to do so. The historical introduction is especially important and suggestive. The committee in their suggestions as to details, that is, in the list of propositions and their classification into those requiring rigorous proof, and those for which only informal proof is desirable, lean away from the side of rigor rather more than seems to me personally to be wise, but this is a question which each teacher must decide for himself. My feeling is based not so much upon consideration of the requirements of the subject as it is upon my experience with students of careless or inexact mental habit. So many students show these qualities in their mathematical work that I shrink from anything which may encourage it. Insistence upon clearly stated geometric demonstrations is a foe to inexactness.

I often wonder whether we do enough in our teaching of mathematics to impress upon our students the surpassing beauty and profound reach of the science. I ought not to say that I wonder whether we do this, because I am well assured that we do not. It is not unusual to hear the pursuit of mathematical study decried as being dry, uninteresting and dehumanizing. It is true that the necessities of the case require that we devote most of our time and effort to mere drill in the mechanical operations, and it is not to be wondered at that students become lost in attention to detail, and fail to get a broad, general view of what it is all about. The teacher standing above, as though on a hill top, can see the distant view, but the student grubbing in what to him is more or less of a jungle, if not a jumble, does not get this view. The teacher should try now and then to give him a glimpse ahead so that he can, to some extent at least, see things in their true perspective. One of the old Greek mathematicians is reported to have said, "The Deity continually geometrizes." This, I take it, was only another way of stating that the whole physical universe "lives and moves and has its being" on a basis of mathematical law. This defines the scope of our science, and if we endeavor to bring our students to an acquaintance with this which is the real incentive to mathematical study and investigation, we shall, I think, be rewarded with a more general interest and with an occasional burst of enthusiasm. I remember reading in a little book describing life at Yale at about the middle of the last century, a little story which has some point in this connection. It was the custom at that time for the different tutors in the College to take turns in leading the services at morning chapel, and upon one occasion, when one of the mathematical tutors was officiating in this capacity, he began his prayer in the following words, "O Thou who guidest the motions of the heavenly bodies as they move in their appointed orbits, the force varying directly as the mass and inversely as the square of the distance," . . . Poor fellow! It was ludicrous enough we must admit but at least we can see that he did not lack an appreciation of the dignity of mathematical principles and their far-reaching importance, although we should probably prefer to use other opportunities of bringing these things to our hearers' attention.

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